Water Wave Generation Due to Initial Disturbance at the Free Surface in an Ocean with Porous Bed

Subhabrata Gangopadhyay, Uma Basu

Abstract – The phenomenon of generation of water waves due to an initial disturbance at the free surface in an ocean with porous bed is investigated within the framework of linear theory. The problem is mathematically formulated as an initial boundary value problem of the potential function. Then the solution of the same is obtained by employing Fourier transform technique. The form of the free surface elevation is arrived at in terms of an infinite integral. Then the method of stationary phase is applied to evaluate this integral approximately for large values of time and distance. The graphs are plotted depicting the variation of free surface elevation against non-dimensionalised values of time and distance for different values of porous parameter.

Keywords – initial boundary value problem, initial disturbance, Fourier transformation, free surface depression, method of stationary phase, porous bed, velocity potential

1 INTRODUCTION

When a nonspherical shock wave caused by a mid-air blast impinges on the surface of a sea, water wave is generated due to an initial surface elevation. The relevant two-dimensional unsteady motion thus formed was studied in the classical treatises of Lamb[1] and Stoker [2] considering linear theory of water waves. They used the method of Fourier transform and obtained the free surface elevation in the form of infinite integrals which were then evaluated approximately for large time and distance from the source of disturbance.

Kranzer and Keller[3] discussed the problem of water wave generated by explosions in case of three-dimensional unsteady motion in water of finite depth due to an initial surface impulse or initial surface elevation on a circular area. They also compared their theory with the available experimental results. Chaudhuri [4] extended these results in case of any initial surface impulse and elevation across arbitrary regions. Wen [5] revisited the extension studied by Chaudhuri and derived asymptotic results using the method of stationary phase as applied to double integrals. Debnath and Guha[6] worked out asymptotic analysis to the integral solutions while investigating the form of the axially symmetric free surface response of an inviscid stratified fluid in presence of an initial displacement of the free surface of the fluid of finite or infinite depth. Mandal [7] considered water waves generated by disturbance at an inertial surface. He employed Laplace transform technique to solve the initial value problem describing waves generated by a disturbance created at the surface of water covered by an inertial surface composed of a thin but uniform disturbance of floating particles. Ghosh et.al[8] examined water wave due to initial disturbance at the inertial surface in a stratified fluid of finite depth. Maity and Mandal [9,10] threw considerable light on the problem of generation of surface waves caused by an initial disturbance at the upper surface of an ocean with an elastic ice cover. They applied Laplace-Fourier transform to solve the problem and derived the elevation of the upper surface and also computed numerical results leading to graphs depicting the variations of surface elevation for large values of time and distance.

In the present paper, the upper surface of the ocean is taken as the free surface while the bed of the ocean is assumed to be porous in nature. The wave motion is generated by an initial disturbance at the free upper surface which is almost concentrated at a point in case of water of finite depth. The disturbance may be thought of as originating in the form of an initial depression at a point on the free surface. Since the motion starts from rest, it is irrotational and can be described by a potential function involving the spatial and time coordinates. The motion in water is two dimensional and remains unsteady. The problem is formulated mathematically as an initial boundary value problem which is then solved analytically by making use of Fourier transform technique.

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The prime concern of our study is to derive the form of the depression of the free surface at any time and distance from the origin of the initial disturbance. The form of the depression of the free surface is obtained as an infinite integral by considering inverse Fourier transform. Finally the infinite integral is approximated by a suitable expression on execution of the asymptotic analysis using stationary phase method in case of a simple form of the initial surface depression.

The novelty involved in the current paper lies in the fact that the natural seepage of the ocean bed is taken into account as the bottom is considered to be permeable unlike the previous authors who considered the bed to be impermeable.

2 MATHEMATICAL FORMULATION

An inviscid, incompressible fluid is taken and the wave motion is generated by some initial disturbance in the neighbourhood of the origin. A two-dimensional rectangular Cartesian coordinate system of axes is considered in which y-axis is taken vertically downwards along the depth of the ocean. The undisturbed free surface corresponds to x -axis, i.e. $y = 0$ and the ocean is of uniform finite depth h below the mean free surface. The motion remains unsteady, two dimensional and irrotational and, therefore, it can be described by a velocity potential $\varphi(x, y, t)$ which satisfies the Laplace equation:

$$\nabla^2\varphi = 0,\ -\infty < x < \infty,\ 0 < y < h,\ t > 0 \quad (1)$$

The free surface boundary conditions are given by:

$$\text{and } \left.\begin{aligned}\varphi_t &= g\eta\\ \varphi_y &= \eta_t\end{aligned}\right\} \text{on } y = 0, -\infty < x < \infty, t > 0 \quad (2)$$

where g is the acceleration due to gravity and $\eta(x, t)$ is the depression of the free surface.

The bottom boundary condition in presence of a porous bed is given by:

$$\varphi_y = G'\varphi \text{ on } y = h \quad (3)$$

where $G' = \frac{\alpha}{\sqrt{\upsilon}}$ is the porous effect parameter. The quantity α is a dimensionless constant which depends on the structure of the porous medium and υ is the permeability of the porous medium.

Since the wave motion is created by an initial depression at the free surface, therefore, the initial conditions of the problem are given by:

$$\left.\begin{aligned}\varphi(x, 0{,}0) &= 0\\ \varphi_t(x, 0{,}0) &= gf(x)\end{aligned}\right\} \text{on } y = 0, t = 0 \quad (4)$$

where $f(x)$ is the initial depression given on the free surface. The form of the depression of the free surface can be obtained at time t as

$$\eta(x, t) = \frac{1}{g}\varphi_t(x, 0, t) \text{ on } y = 0 \quad (5)$$

3 METHOD OF SOLUTION

The Fourier transformation technique is employed with respect to the spatial variable to solve the initial boundary value problem (1)-(5).

The Fourier transformations of $\varphi(x, y, t)$ and $\eta(x, t)$ are given by:

$$\left.\begin{aligned}F(\varphi) = \overline{\varphi}(x, y, t) &= \int_{-\infty}^{\infty} e^{ikx}\varphi(x, y, t)dx\\ &\ and\\ F(\eta) = \overline{\eta}(x, t) &= \int_{-\infty}^{\infty} e^{ikx}\eta(x, t)dx\end{aligned}\right\} \quad (6)$$

Using (6) the IBVP (1) – (4) reduce to:

$$\overline{\varphi}_{yy} - k^2\varphi = 0 \text{ on } 0 < y < h, t > 0 \quad (7)$$

$$\left.\begin{aligned}\overline{\varphi}_t &= g\overline{\eta}\\ \overline{\varphi}_y &= \overline{\eta}_t\end{aligned}\right\} \text{on } y = 0, t > 0 \quad (8)$$

The bottom boundary condition becomes

$$\overline{\varphi}_y = G'\overline{\varphi} \text{ on } y = h \quad (9)$$

The initial conditions become

$$\left.\begin{aligned}\overline{\varphi}(k, 0{,}0) &= 0\\ \overline{\varphi}_t(k, 0{,}0) &= gF(k)\end{aligned}\right\} \text{on } y = 0, t = 0 \quad (10)$$

where $F(k)$ is the Fourier transformation of the function $f(x)$.

Equation $\overline{\varphi}_{yy} - k^2\varphi = 0$ implies

$\overline{\varphi}(k, y, t) = c_1 coshky + c_2 sinhky$, where the unknown constants c_1 and c_2 are to be found out using the boundary and initial conditions.

Using (9) we get,

$$c_1 ksinhkh + c_2 kcoshkh = G'c_1 coshkh + G'c_2 sinhkh$$

or, $ksinhkh - G'coshkh = \left(\frac{c_2}{c_1}\right).(G'sinhkh - kcoshkh)$

Therefore, $\overline{\varphi}(k, y, t) = c_1\left[coshky + \left(\frac{c_2}{c_1}\right)sinhky\right]$

$$= \frac{c_1}{G'sinhkh - kcoshkh}\cdot[G'sinhk(h - y) - kcoshk(h - y)]$$

$$= \overline{A}(k, t)[G'sinhk(h - y) - kcoshk(h - y)] \quad (11)$$

where $\overline{A}(k, t) = \frac{c_1}{G'sinhkh - kcoshkh}$

Now, using (8) we get,

$$\frac{d\overline{\eta}}{dt} = -\overline{A}(k, t)[G'kcoshkh - k^2 sinhkh]$$

$$\frac{d\overline{A}}{dt} = \frac{g\overline{\eta}}{G'sinhkh - kcoshkh}$$

$$\overline{A}_t(k, 0) = \frac{gF(k)}{G'sinhkh - kcoshkh}$$

Now, $\frac{d^2\overline{A}}{dt^2} = \frac{gk}{G'sinhkh - kcoshkh}\cdot\frac{d\overline{\eta}}{dt}$

$= -gk \cdot \frac{G'coshkh - ksinhkh}{G'sinhkh - kcoshkh} \cdot \overline{A}$

or, $\frac{d^2\overline{A}}{dt^2} = -\mu^2 \overline{A}$ where $\mu^2 = gk \cdot \frac{G' - ktanhkh}{G'tanhkh - k}$ (12)

We seek $\overline{A}(k,t)$ in the form $\overline{A}(k,t) = c_3 cos\mu t + c_4 sin\mu t$ where the constants c_3 and c_4 are determined as follows:

$\overline{A}(k,0) = 0$ gives $0 = c_3 cos\mu t$ which implies $c_3 = 0$.

$\overline{A}_t = c_4 \mu cos\mu t$ which implies $\overline{A}_t(k,0) = c_4 \mu$

Therefore, $c_4 \mu = \frac{gF(k)}{G'sinhkh - kcoshkh}$.

Hence, $\overline{A} = \frac{gF(k)}{\mu(G'sinhkh - kcoshkh)} \cdot sin\mu t$

Thus, $\overline{\varphi}(k,y,t) = \frac{gF(k)sin\mu t}{\mu} \cdot \frac{G'sinhk(h-y) - kcoshk(h-y)}{G'sinhkh - kcoshkh}$

Now, considering inverse Fourier transformation we obtain the following infinite integral representation of the potential function:

$$\varphi(x,y,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \cdot \frac{gF(k)sin\mu t}{\mu} \cdot \frac{G'sinhk(h-y) - kcoshk(h-y)}{G'sinhkh - kcoshkh} dk$$

and the form of the depression of the free surface is obtained as:

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(k) cos\mu t dk \quad (13)$$

4 ASYMPTOTIC ANALYSIS: STATIONARY PHASE APPROXIMATION

We consider the wave motion to be generated by an initial depression of the free surface concentrated at the origin. Hence, we take $f(x) = \delta(x)$ where $\delta(x)$ stands for Dirac's delta function. In this case, $(k) = \frac{1}{\sqrt{2\pi}}$. The representation of the surface elevation (13) now reduces to:

$\eta(x,t) = \frac{1}{\pi} \int_0^{\infty} coskxcos\mu t dk$ where μ is given by (12).

This is an oscillatory integral which can be approximated by employing the method of stationary phase for large x and t such that $\frac{x}{t}$ remains finite.

We can write, $\eta(x,t) = \frac{1}{2\pi} Re\left[\int_0^{\infty} \left(e^{itS(k)} + e^{itS(-k)}\right)\right] dk$ (14)

where $S(k) = \mu - k \cdot \frac{x}{t} = \sqrt{\frac{gk(G' - ktanhkh)}{G'tanhkh - k}} - k \cdot \frac{x}{t}$.

The second term of the integral (14) has no stationary point within the range of integration. The stationary point of the first integral is given by the following equation:

$S'(k) = \frac{ds}{dk} = 0$. (15)

In view of the fact that the function $S'(k)$ is monotone decreasing in nature, the stationary point k_0 is the unique positive real root of the transcendental equation (15) and the integral (13) representing the free surface elevation can be approximated according to the stationary phase method as

$$\eta(x,t) \cong \frac{1}{\sqrt{2\pi t\,|S''(k_0)|}} \cdot cos\left[t\,S(k_0) - \frac{\pi}{4}\right] \quad (16)$$

5 GRAPHS

We consider the case when the initial disturbance is applied in the form of initial depression at the origin by taking $f(x) = \delta(x)$. In this case, the approximated form of the free surface depression is given by the expression (16). The physical quantities are non-dimensionalised as follows:

$x' = \frac{x}{h}$; $y' = \frac{y}{h}$; $t' = \sqrt{\frac{g}{h}}t$ and $\eta' = \frac{\eta}{h}$.

In fig. 1, $\eta'(x,t)$ is plotted against $50 \le x' \le 100$ for $G'h = 0.2, 0.3, 0.35$ while t' is fixed at $t' = 150$. It is noticed that the height of oscillation of η' gradually increases with increase in x'. The height of oscillation increases with increase in the porous parameter $G'h$.

In fig. 2, $\eta'(x,t)$ is plotted against $115 \le t' \le 150$ for $G'h = 0.2, 0.3, 0.35$ while x' is fixed at $x' = 100$. It is noticed the height of oscillation of η' gradually increases with increase in t'. It is further observed that, in this case, unlike in fig. 1, the height of oscillation decreases with increase in the porous parameter $G'h$.

The effect of seepage is more pronounced at relatively smaller values of t'.

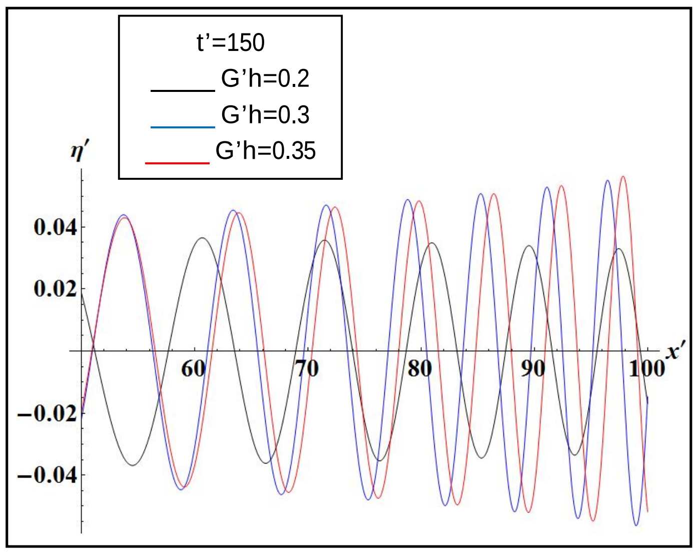
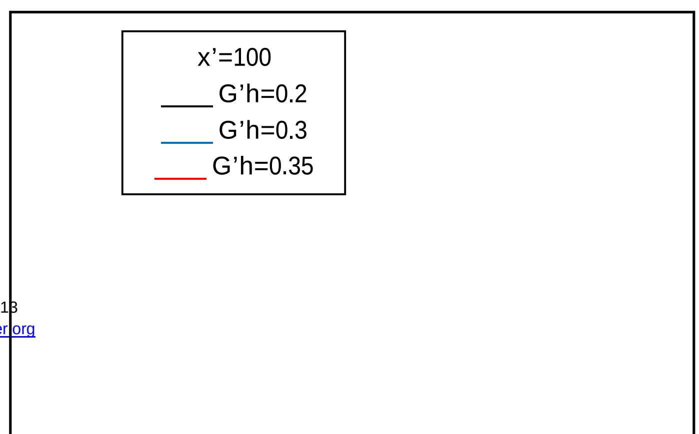


Fig. 1 Surface elevation plotted against distance at a fixed time for different values of porous parameter



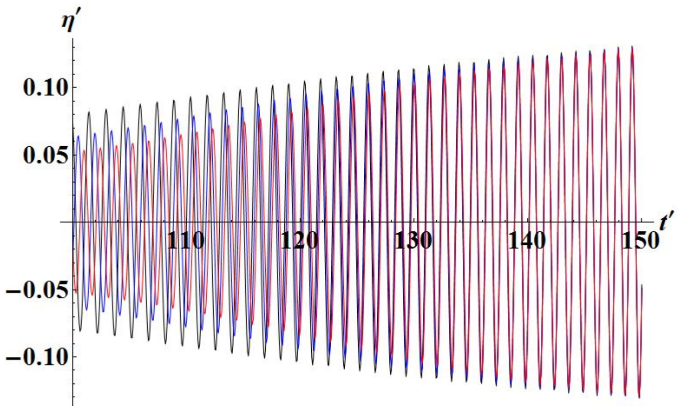


Fig. 2 Surface elevation plotted against time at a fixed distance for different values of porous parameter

6 CONCLUSION

The problem of water wave generation due to an initial disturbance at the free surface of the ocean with uniform finite depth is investigated using linear theory of water waves. The Fourier transformation technique is employed to solve the problem analytically in finding out integral representation of the potential function and surface elevation after formulating the problem as an initial boundary value problem. The integral representation of the surface elevation is approximated by means of stationary method analysis. The sea bed is considered to be porous rather than an impermeable one. The effect of seepage in the bottom of the ocean is significant as it is clear from the graphs.

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